Journal of Mathematical Cognition and Instruction
ISSN 0000-0000
Vol. 01, No. 01, 2026, 01-17
DOI: https://doi.org/10.31949/jmci.v1i1.17913

Epistemic alignment in students' understanding of rational inequalities: a didactical design research beyond the cognition perspective

Cantika Fitriyani*
Department of Mathematics Education, Universitas Majalengka, Majalengka, Indonesia
***Corresponding Author:** cantikafitriyani@email.com

Article History
Received: 01-02-2026
Revised: 20-03-2026
Accepted: 09-04-2026

Keywords
Epistemic alignment,
Beyond cognition,
Rational inequalities,
Didactical design research,
Conceptual understanding

ABSTRACT

This study investigates students' understanding of rational inequalities from a beyond-cognition perspective by examining how epistemic alignment develops as structural complexity increases. Moving beyond traditional cognitive and error-analysis approaches, this research conceptualizes learning as an epistemic process shaped by the interaction between students' reasoning, task design, and institutional mathematical norms. The study employed a Didactical Design Research (DDR) approach within a qualitative interpretative framework, involving Grade 10 students in a public senior high school in Indonesia. Data were collected through diagnostic tasks, classroom observations, semi-structured interviews, and document analysis, and analyzed using an epistemic lens grounded in concept image theory and institutionalization. The findings reveal a systematic redistribution of students' responses across three increasingly complex tasks. Integrated understanding decreased significantly, while procedural and fragmented responses increased. This pattern indicates that students' understanding is not stable but dynamically reorganized in response to structural demands. Three levels of epistemic alignment were identified: integrated (coherent and transferable reasoning), procedural (compensatory reasoning without conceptual grounding), and fragmented (structural misalignment and breakdown of coordination). The study contributes to the literature by reconceptualizing students' errors as indicators of epistemic misalignment rather than cognitive deficiencies, and by proposing a data-driven model of epistemic alignment. These findings highlight the importance of designing instructional tasks that support conceptual–procedural integration and suggest that meaningful mathematical understanding emerges through the coordination of multiple epistemic dimensions.

INTRODUCTION

Mathematics education is increasingly understood as a complex and situated process that extends beyond the boundaries of individual cognition. Rather than being confined to internal mental processes, learning emerges through dynamic interactions among learners, teachers, and the instructional environment in which mathematical meaning is constructed and negotiated. This perspective is aligned with socio-constructivist views that emphasize the role of social interaction, affect, and context in shaping mathematical understanding (Gravemeijer, 2020; Eynde et al., 2006). More recent developments further extend this view by conceptualizing learning as an epistemic process that integrates cognitive, affective, and institutional dimensions (Chiu et al., 2026). Consequently, knowledge is not merely acquired but progressively shaped through engagement with structured tasks, symbolic representations, and socially mediated validation processes, thereby requiring a shift toward a more integrated analytical lens that connects cognition, didactical design, and institutional norms.

Within the Theory of Didactical Situations (TDS), learning is conceptualized as a sequence of interconnected phases—action, formulation, validation, and institutionalization—through which students' informal constructions evolve into mathematically legitimate knowledge (Novotná & Hošpesová, 2022; Bahn & Winsløw, 2019). Among these phases, institutionalization occupies a critical position, as it represents the moment when personal meanings are reorganized and aligned with established mathematical structures. However, emerging studies suggest that institutionalization should not be reduced to a discrete instructional phase, but rather understood as a continuous epistemic mechanism that mediates the relationship between students' concept images and formal concept definitions (Tall & Vinner, 1981). From this perspective, institutionalization serves as a bridge between individual reasoning and the broader socio-didactical system in which knowledge is legitimized.

This perspective becomes particularly relevant when considering the transformation of mathematical knowledge through didactical transposition. Within the Anthropological Theory of the Didactic (ATD), knowledge undergoes systematic recontextualization as it moves from scholarly mathematics to classroom instruction (Abou-Hayt, 2024). However, this transformation is inherently selective and often reduces conceptual meaning to procedural techniques, echoing earlier distinctions between procedural and conceptual knowledge (Greeno, 1978). As a consequence, students frequently engage with tasks that are operationally accessible but epistemologically fragmented, leading to structural misalignments between intended knowledge, instructional practices, and students' understanding.

Empirical studies have consistently shown that students' difficulties in mathematics are frequently associated with discrepancies between concept image and concept definition (Tall & Vinner, 1981). However, much of this research has approached these discrepancies primarily from a cognitive or error-analysis perspective, focusing on procedural mistakes or misconceptions (Abakah & Brijlall, 2024; Hoth et al., 2022). While such approaches provide valuable descriptive insights, they offer limited explanatory power regarding how these discrepancies emerge within broader didactical and institutional contexts. In particular, they tend to overlook the role of instructional design and institutional mediation in shaping the epistemic conditions under which understanding develops (Usó Doménech et al., 2022).

From a beyond-cognition standpoint, students' errors should not be interpreted solely as failures of reasoning, but rather as indicators of incomplete or uneven institutionalization. This interpretation is supported by recent studies highlighting that students' responses reflect underlying epistemic beliefs and positioning toward mathematical knowledge (Richter & Schmid, 2010; Alvidrez et al., 2024). In this view, different forms of responses—whether correct with justification, correct without justification, or incorrect—represent varying degrees of epistemic alignment between personal understanding and institutional mathematical knowledge. Thus, learning difficulties can be more accurately understood as manifestations of how students operate within different epistemic frameworks shaped by prior experience, instructional practices, and task structures.

This issue is particularly evident in the context of rational inequalities, which require the coordination of multiple conceptual dimensions, including algebraic manipulation, domain restrictions, and sign analysis. The complexity of these interactions demands not only procedural fluency but also coherent conceptual integration. However, existing studies have largely focused on identifying common procedural errors without sufficiently examining how these errors relate to the quality of institutionalization embedded in instruction. Addressing this gap requires a shift in analytical perspective—from viewing learning as an individual cognitive process to understanding it as an emergent phenomenon shaped by the interaction between cognition, didactical design, and institutional structures. In this regard, Didactical Design Research (DDR) provides a robust methodological framework, as it enables systematic investigation of how instructional interventions mediate students' conceptual development through structured didactical situations.

Accordingly, the present study aims to investigate how varying levels of institutionalization are manifested in students' understanding of rational inequalities, and how these levels reflect the extent to which conceptual and procedural knowledge are integrated. Specifically, this study positions institutionalization not merely as a pedagogical phase, but as a cognitive–epistemological mechanism that operates beyond cognition, linking individual reasoning with institutional mathematical structures. By doing so, this study contributes to the literature in two significant ways. First, it reconceptualizes students' errors as indicators of epistemic misalignment rather than isolated cognitive failures. Second, it offers a more comprehensive framework for understanding how meaningful mathematical understanding emerges through the alignment of concept image, instructional design, and institutional norms.

METHODS

Research Design

This study employed a Didactical Design Research (DDR) approach situated within a qualitative interpretative paradigm. The adoption of a qualitative approach is grounded in its ability to capture the complexity of mathematical learning as a situated, meaning-making process rather than as a set of measurable variables (Bikner-Ahsbahs et al., 2015). Within mathematics education research, qualitative inquiry enables an in-depth exploration of how students construct, negotiate, and transform mathematical understanding through interaction with tasks, representations, and instructional mediation. The selection of DDR was not merely methodological but also epistemological. DDR provides a systematic framework for analysing the relationship between instructional design and students' conceptual development through iterative cycles of design, implementation, and retrospective analysis. This approach has been shown to be particularly effective in examining how didactical configurations shape students' reasoning processes and learning trajectories (Jatisunda et al., 2025). In alignment with a beyond-cognition perspective, this study conceptualises mathematical understanding as an emergent phenomenon arising from the interaction between students' reasoning, instructional design, and institutional mathematical norms. Accordingly, institutionalisation is not positioned as a terminal instructional phase but as a continuous epistemic mechanism through which students' informal concept images are progressively reorganised and aligned with formal mathematical structures. This positioning requires a methodological approach that is sensitive to both observable classroom interactions and underlying epistemic processes, which qualitative interpretative research is well-suited to capture (McCollum, 2022).

Research Context and Participants

The study was conducted from February to April 2024 in a public senior high school in Majalengka Regency, Indonesia. The participants were Grade 10 students engaged in learning rational inequalities, a topic characterised by the coordination of multiple conceptual dimensions, including algebraic manipulation, domain restriction, and sign analysis. In addition to students, the study involved a mathematics teacher and a group of expert practitioners consisting of mathematics educators, researchers, and experienced teachers. These experts participated through Focus Group Discussions (FGDs) to support the validation and refinement of the didactical design and its interpretation. The inclusion of multiple actors reflects the understanding that mathematical learning is socially and institutionally situated, and therefore must be analyzed through multiple perspectives (Quane, 2025).

Research Procedure

The research followed the three iterative phases of DDR: preliminary analysis, didactical design and implementation, and retrospective analysis. These phases enable a systematic investigation of how instructional interventions mediate students' conceptual development while capturing the dynamic nature of classroom learning. In the preliminary analysis, document analysis was conducted to examine how rational inequalities are represented in textbooks and instructional materials. Document analysis is widely recognized as a rigorous qualitative method for identifying patterns of representation and underlying assumptions within educational materials (Bowen, 2009). This phase also included discussions with teachers to identify potential learning obstacles, which were interpreted as indicators of epistemic misalignment rather than merely cognitive difficulties.

In the didactical design and implementation phase, instructional tasks were designed to progressively elicit, challenge, and reorganize students' reasoning. The design emphasized transitions from procedural engagement toward conceptual integration, reflecting the need to capture how students' understanding evolves through structured learning experiences. Classroom observations were conducted to document real-time interactions, as observational methods are essential for capturing how mathematical meaning is enacted and negotiated in situ (Quane, 2025). In the retrospective analysis, data from observations, student work, and interviews were interpreted to trace the development of students' reasoning across tasks. This phase aligns with qualitative analytical traditions that emphasize meaning reconstruction and interpretation of learning processes rather than outcome measurement (Bikner-Ahsbahs et al., 2015).

Data Collection

Data were collected using multiple complementary methods to ensure depth and triangulation:

1. Document analysis to examine instructional materials and identify forms of didactical transposition (Bowen, 2009)
2. Diagnostic tasks to capture students' evolving conceptual understanding
3. Semi-structured interviews to explore students' reasoning processes
4. Classroom observations to document interactional dynamics in learning (Quane, 2025)
5. Focus Group Discussions (FGDs) to validate interpretations and refine analytical perspectives

The use of multiple data sources reflects a commitment to capturing learning as a multidimensional and situated process, consistent with qualitative research principles in mathematics education (Bikner-Ahsbahs et al., 2015).

Data Analysis

Data analysis was conducted using an interpretative qualitative approach grounded in concept image theory, while explicitly integrating institutionalisation as an analytical lens. Qualitative analysis in mathematics education emphasises the identification of patterns in reasoning, meaning-making, and representation, rather than numerical generalisation (Bikner-Ahsbahs et al., 2015). All data sources were systematically coded to identify how procedural actions were coordinated with conceptual understanding. This approach aligns with qualitative analytical frameworks that seek to uncover underlying structures of reasoning and epistemic organisation (McCollum, 2022). To operationalise institutionalisation, students' responses were analysed across four dimensions:

1. Conceptual articulation
2. Procedural–conceptual integration
3. Justification and validation
4. Alignment with formal definition

These dimensions enabled the categorisation of students' responses into levels of epistemic alignment, providing a nuanced interpretation of learning beyond surface-level correctness.

Credibility and Trustworthiness

To ensure methodological rigour, several strategies were employed:

1. Data triangulation across multiple sources
2. Member checking with participants
3. Audit trail to document analytical decisions
4. Peer debriefing with expert practitioners

The use of an audit trail is particularly important in qualitative research, as it enhances transparency and allows the research process to be systematically traced and evaluated (Bowen, 2009). These strategies collectively ensure that the findings accurately reflect the complexity of learning as a situated and epistemic process.

RESULTS AND DISCUSSION

Results

Task Structure and Analytical Orientation

To systematically examine how students' understanding develops as structural complexity increases, this study employed a sequence of three tasks that progressively integrate multiple conceptual dimensions of rational inequalities. The design of these tasks was not arbitrary; rather, each task was constructed to foreground specific epistemic demands, enabling the analysis to capture how students coordinate conceptual and procedural elements under varying levels of cognitive–structural load. From an analytical perspective, the sequencing of tasks serves as a **didactical instrument for diagnosing epistemic alignment**, allowing the study to trace not only whether students arrive at correct solutions, but how their reasoning adapts when additional conceptual constraints are introduced.

Task 1: Basic Structure (Single-Layer Coordination)

Task 1 focuses on fundamental sign analysis and domain awareness within a relatively stable structural context. At this level, students are required to:

1. Identify critical points (zeros and discontinuities)
2. Recognize domain restrictions
3. Partition the number line into intervals
4. Determine the sign of the expression across intervals

The epistemic demand of this task is single-layer coordination, in which students operate within a relatively bounded structure that does not require them to transform expressions. Conceptual understanding is primarily reflected in the ability to connect algebraic structure (numerator–denominator relationship) with sign behavior. From an analytical standpoint, Task 1 serves as a baseline indicator of epistemic coherence, revealing whether students exhibit a stable integration of procedural actions (e.g., interval testing) and conceptual meaning (e.g., why sign changes occur). At this stage, correct responses with justification indicate that students can operate within an already structured mathematical framework without needing reconfiguration.

Task 2: Transformation Structure (Dual Coordination)

Task 2 introduces an additional layer of complexity by requiring students to transform the given inequality into a standard rational form before applying sign analysis. This transformation process demands:

1. algebraic manipulation to unify expressions
2. reinterpretation of the inequality structure
3. coordination between transformation and subsequent reasoning steps

The epistemic demand shifts from single-layer to dual coordination, where students must simultaneously manage transformation processes and interpret the resulting structure. Unlike Task 1, where the structure is given, Task 2 requires students to actively construct the structure on which further reasoning depends.

This introduces a critical analytical dimension: structural reconstruction. Students must not only apply known procedures but also understand how transformations preserve or alter mathematical meaning.

Empirically, this task serves to identify whether students' understanding is:

1. structurally grounded (able to maintain meaning through transformation), or
2. procedurally dependent (able to perform manipulation without understanding its implications)

Thus, Task 2 functions as a transition point, revealing the extent to which students' reasoning can remain coherent when the problem requires reorganisation of its mathematical form.

Task 3: Integrated Structure (Multi-Layer Coordination)

Task 3 represents the highest level of structural complexity, requiring students to coordinate multiple conceptual dimensions simultaneously. In this task, students must:

1. Transform the inequality into a rational expression
2. Identify domain restrictions and discontinuities
3. Analyze sign behaviour across intervals
4. Integrate linear comparison with rational structure
5. Justify the solution in relation to inequality conditions

The epistemic demand here is multi-layer coordination, where several interdependent components must be synchronised within a single reasoning process. Unlike Task 2, where transformation precedes analysis, Task 3 requires continuous coordination between transformation, interpretation, and validation.

This level introduces epistemic integration under constraint, where:

1. Domain restrictions interact with algebraic structure
2. Sign analysis must be interpreted in relation to the inequality direction
3. Procedural steps must be justified within a coherent conceptual framework

From an analytical perspective, Task 3 serves as a stress test of epistemic stability, revealing whether students' understanding is robust enough to sustain coherence when multiple conceptual elements are activated simultaneously.

Students who succeed in this task demonstrate:

1. structurally consistent reasoning
2. transferability of conceptual knowledge
3. alignment with formal mathematical definitions

Conversely, failure in this task often indicates:

1. fragmentation of concept image
2. breakdown in procedural–conceptual integration
3. reliance on inappropriate prior schemas

Cross-Task Analytical Function

Taken together, the three tasks form a progressive epistemic gradient, enabling the study to observe how students' understanding evolves across increasing levels of structural demand:

Table 1. Epistemic Demands and Analytical Functions of Task Design Across Levels of Structural Complexity

Task	Epistemic Demand	Analytical Function
Task 1	Single-layer coordination	Baseline of conceptual–procedural coherence
Task 2	Dual coordination (transformation + analysis)	Detection of structural reconstruction ability
Task 3	Multi-layer coordination	Measurement of epistemic stability under complexity

This structured progression allows the analysis to move beyond static evaluation of correctness toward a dynamic examination of how understanding is constructed, reorganized, and sometimes destabilized.

Table 2. Distribution of Student Responses Across Tasks

Category of Understanding	Task 1	Task 2	Task 3
Integrated (Correct + Justification)	19 (68%)	15 (54%)	12 (21%)
Procedural (Correct without Justification)	4 (14%)	5 (18%)	18 (32%)
Fragmented (Incorrect)	5 (18%)	8 (29%)	26 (46%)

Table 2. presents the distribution of students' responses across three tasks with increasing structural complexity. A clear pattern emerges: the proportion of students demonstrating integrated understanding decreases progressively from 68% in Task 1 to 21% in Task 3, while the proportion of students providing fragmented responses increases substantially from 18% to 46%. Procedural responses also show a notable rise, particularly in Task 3. This redistribution suggests that students' understanding is not stable across tasks, but shifts in response to increasing conceptual demands. Specifically, as tasks require the coordination of multiple mathematical structures—such as transformations, domain restrictions, and sign analysis—students' reasoning tends to shift away from conceptually integrated forms toward either procedural reliance or structural inconsistency. Importantly, the increase in procedural responses indicates that correctness can be maintained even when conceptual understanding weakens. However, this form of reasoning appears to function as a compensatory mechanism rather than a stable form of understanding. In contrast, the sharp increase in fragmented responses suggests that higher task complexity exposes underlying weaknesses in students' conceptual structures, leading to a breakdown in reasoning. The data indicate that students' understanding is highly sensitive to structural complexity and that the ability to maintain integrated reasoning depends on the extent to which conceptual and procedural elements are coherently coordinated.

Empirical Pattern: Redistribution of Epistemic Alignment

Table 1 reveals a systematic redistribution of students' responses as task complexity increases. The proportion of integrated understanding decreases progressively from 68% in Task 1 to 21% in Task 3, while the proportion of fragmented responses increases substantially from 18% to 46%. Procedural responses also show a notable rise, particularly in Task 3, indicating a shift in the dominant form of reasoning. Importantly, this pattern does not merely indicate a decline in performance; rather, it reflects a reconfiguration of epistemic alignment under increasing structural demand. As tasks require the coordination of multiple conceptual dimensions—such as transformation, domain restriction, and relational sign analysis—students' reasoning shifts from structurally coherent forms toward either procedure-dominant strategies or structurally inconsistent interpretations.

This redistribution suggests the presence of an adaptive but unstable reasoning mechanism. When conceptual–procedural integration weakens, students tend to rely on familiar procedures as a compensatory strategy. However, when such compensation is insufficient—particularly in tasks requiring multi-layer coordination—reasoning breaks down into fragmented forms that reflect misinterpretation of underlying mathematical structures. Therefore, the data indicate that understanding is not a stable attribute, but a situated and dynamically reorganized configuration, contingent upon the level of structural integration required by the task. In this sense, task complexity functions as an epistemic filter, revealing not only differences in correctness but differences in the stability, coherence, and transferability of students' reasoning. The nature of this reconfiguration becomes more evident when examined through qualitative evidence of students' reasoning.

Integrated Understanding: Epistemic Coherence Under Structural Constraint

Students classified within the integrated category demonstrate a high degree of epistemic coherence, characterized by the alignment of conceptual articulation, procedural reasoning, and justification. Importantly, this group maintains structurally consistent reasoning even as task complexity increases, although its proportion declines.

This coherence is empirically illustrated in the following explanation:

“The denominator cannot be zero, so I excluded that value. Then I found the points where the expression changes sign and tested each interval to see where it becomes negative.”

This response reflects a fully coordinated reasoning process in which:

1. Domain restriction is explicitly articulated as a structural condition
2. Sign analysis is employed as a relational tool rather than a procedural routine
3. Interval testing is justified within an integrated conceptual framework

Crucially, these elements are not applied sequentially but are synchronized within a unified reasoning structure, indicating that students can coordinate multiple conceptual dimensions simultaneously. This coordination suggests that their concept images have been reorganized into forms that are compatible with formal mathematical definitions. From an analytical perspective, this group exhibits epistemic stability, where reasoning remains invariant under increasing structural demands. However, the decline in its proportion (from 68% to 21%) indicates that such stability is both cognitively and structurally demanding. It requires not only the presence of conceptual and procedural knowledge, but their real-time integration under conditions of increasing complexity. Thus, integrated understanding can be interpreted not merely as correct performance but as evidence of a highly consolidated epistemic configuration, in which reasoning is transferable, structurally grounded, and resistant to task-induced disruption**.**

Procedural Understanding: Compensatory Mechanism Under Increasing Complexity

The increase in procedural responses (from 14% to 32%) suggests a shift toward procedure-dominant reasoning as task complexity increases. While students in this category arrive at correct answers, their reasoning lacks conceptual grounding and explicit justification.

This pattern is reflected in the following statement:

“I tested numbers in the intervals and found the answer, but I’m not sure why that interval works.”

This explanation indicates that:

1. Procedural execution remains intact
2. Conceptual articulation is absent
3. Justification is replaced by empirical verification

Crucially, this pattern reflects a decoupling between procedural action and conceptual meaning, in which students can execute solution steps without accessing the underlying structural relationships that justify them. Rather than coordinating conceptual and procedural knowledge, students rely on surface-level operational routines. Analytically, this category represents a transitional epistemic state, in which procedural knowledge functions as a compensatory mechanism in response to weakening conceptual integration. As task complexity increases and requires multi-layer coordination, students appear to stabilise their reasoning by reverting to familiar procedures that can be applied with minimal conceptual processing. However, this compensatory strategy is inherently limited. Because it is not grounded in structural understanding, it lacks epistemic coherence and transferability, making it vulnerable to breakdown when procedural cues are insufficient or when tasks deviate from familiar formats.

Thus, procedural understanding can be interpreted not as a stable form of knowledge, but as an adaptive yet fragile configuration, situated between integrated reasoning and structural misalignment.

Fragmented Understanding: Structural Breakdown and Epistemic Misalignment

The most significant shift occurs in the fragmented category, which increases from 18% in Task 1 to 46% in Task 3. Unlike procedural responses, this category reflects systematic structural misinterpretation, rather than incomplete or partially developed reasoning.

This is evidenced in the following explanation:

"I solved the numerator and denominator separately because both have to be negative."

This response reveals:

1. a fundamental misinterpretation of the relational structure of rational expressions
2. absence of coordination between the numerator and the denominator as an integrated system
3. reliance on inappropriate prior schemas, particularly linear reasoning applied in a non-linear context

Crucially, this pattern reflects not a lack of reasoning but a misdirected coherence, in which students construct internally consistent explanations that are incompatible with formal mathematical structures. In this sense, their reasoning is not random but systematically organized around incorrect structural assumptions. From an analytical standpoint, these responses indicate epistemic misalignment, in which students' concept images are stabilized within personal or previously learned frameworks that do not align with institutional mathematical knowledge. This misalignment is particularly evident in the breakdown of relational reasoning, which is essential for understanding rational inequalities as interconnected expressions rather than independent components. Importantly, the sharp increase in this category suggests that higher task complexity not only increases difficulty but also activates latent inconsistencies within students' conceptual frameworks. When tasks require multi-layer coordination—such as integrating transformation, domain restriction, and sign analysis—students who lack structurally grounded understanding are unable to reorganize their reasoning accordingly. As a result, their responses collapse into fragmented forms, characterized by inappropriate procedures and a lack of conceptual integration. This indicates that fragmented understanding represents not just low performance but a structural breakdown of epistemic coordination, in which previously functional reasoning strategies become incompatible with the task's demands.

Cross-Task Synthesis: Epistemic Stability as a Function of Structural Complexity

By integrating quantitative distribution (Table 1) with qualitative evidence from students' explanations, a coherent cross-task progression becomes evident. This progression reflects not only changes in performance but a systematic transformation in the structure of students' reasoning as task complexity increases. In Task 1 (low complexity), students predominantly demonstrate integrated understanding, indicating a relatively high level of epistemic stability. At this level, reasoning remains coherent because the task requires only single-layer coordination, allowing students to align conceptual and procedural elements within a stable structure.

In Task 2 (moderate complexity), this stability begins to weaken, giving rise to a transitional phase characterized by increasing procedural reliance. As the task introduces transformation processes, students are required to reconstruct the mathematical structure before applying familiar procedures. This additional demand disrupts the coordination between conceptual meaning and procedural execution, leading students to rely more heavily on operational routines as a stabilizing strategy. In Task 3 (high complexity), epistemic instability becomes dominant, as evidenced by the sharp increase in fragmented responses. At this level, students must coordinate multiple interdependent conceptual dimensions simultaneously. When this multi-layer coordination fails, reasoning shifts from partially coherent (procedural) to structurally inconsistent (fragmented), indicating a breakdown in epistemic organization.

This progression suggests that epistemic stability is not an inherent property of students' knowledge but a function of structural complexity and the demands of coordination. Specifically, the ability to maintain integrated understanding depends on whether students can synchronize multiple conceptual elements in real time. When this coordination is disrupted, students' reasoning follows two distinct pathways: procedural compensation, in which familiar routines are used to maintain correctness, or structural breakdown, in which reasoning becomes incompatible with formal mathematical structures. Thus, task complexity operates as a diagnostic gradient of epistemic alignment, revealing not only differences in correctness but the underlying stability, coherence, and adaptability of students' reasoning across contexts.

Data-Driven Model of Epistemic Alignment

The combined analysis of response distribution and reasoning patterns supports the development of a three-level empirical model of students' epistemic alignment. This model does not merely classify students' responses but also captures systematic variations in how mathematical understanding is structured, stabilised, and transformed as task complexity increases.

Table 3. A Three-Level Empirical Model of Students' Epistemic Alignment

Level	Empirical Indicator	Epistemic Characteristic
Integrated	Decreases with complexity	Structurally coherent, justified, and transferable reasoning
Procedural	Increases in moderate–high complexity	Compensatory procedural reasoning with limited conceptual grounding
Fragmented	Dominant in high complexity	Structural misalignment and breakdown of epistemic coordination

Importantly, these levels should not be interpreted as fixed categories, but as positions along a continuum of epistemic alignment. Students may shift between levels depending on the structural demands of the task, indicating that understanding is dynamically reorganised rather than statically possessed.

From an analytical perspective, each level reflects a distinct mode of coordination between conceptual and procedural knowledge:

1. Integrated understanding represents full epistemic alignment, where reasoning is coherent, justified, and resilient across contexts.
2. Procedural understanding reflects partial alignment, where procedural fluency is maintained but decoupled from conceptual meaning, functioning as a compensatory mechanism.
3. Fragmented understanding indicates epistemic misalignment, where reasoning is internally consistent but structurally incompatible with formal mathematical knowledge.

Crucially, the transition across these levels is governed by the increasing demand for multi-layer conceptual coordination. As task complexity intensifies, students who are unable to sustain this coordination shift from integrated reasoning toward procedural compensation, and ultimately toward structural breakdown. Thus, this model provides an empirical basis for reconceptualising students' errors and correct responses not as binary outcomes, but as indicators of varying degrees of epistemic alignment. In this sense, correctness alone is insufficient to capture the quality of understanding; rather, what matters is the stability, coherence, and transferability of the reasoning underlying the response.

Discussion

The findings of this study indicate that mathematical understanding is more appropriately conceived as an epistemic configuration rather than a stable cognitive attribute. The observed shift from integrated to procedural and subsequently fragmented responses reflects not merely a decline in performance, but a transformation in how students organise, justify, and stabilise mathematical knowledge under varying structural demands . In this sense, understanding emerges through the interplay between task structure, instructional mediation, and the institutional norms that define what counts as legitimate mathematical knowledge (Brown & Clarke, 2012; Castela & Romo-Vázquez, 2022).

Within the framework of the Theory of Didactical Situations, these findings invite a reconsideration of institutionalisation. Rather than functioning as a discrete terminal phase, institutionalisation appears to operate as a variable and ongoing epistemic process. Students who demonstrate integrated understanding exhibit forms of stabilised reasoning in which their concept images are reorganised in alignment with formal mathematical structures (Novotná & Hošpesová, 2022). Yet the marked reduction of this group in more complex tasks suggests that such stabilisation is contingent rather than permanent, dependent on the capacity to coordinate multiple conceptual dimensions simultaneously. This reinforces the view that epistemic quality in school mathematics is not determined solely by access to procedures, but by the extent to which conceptual relations can be sustained and articulated across contexts (Hudson, 2018).

The increase in procedural responses points to a different dynamic. Students continue to arrive at correct solutions, yet without the capacity to justify their reasoning in conceptual terms. This pattern reflects a separation between procedural execution and conceptual meaning, a distinction long recognised in the literature (Greeno, 1978; McCormick, 1997). In the present context, however, procedural reasoning appears not simply as a deficit, but as an adaptive response. When conceptual coordination weakens, students rely on familiar procedures as a means of maintaining operational control. Such behaviour is consistent with research on epistemic cognition, which shows that learners adjust their strategies in relation to task demands and their beliefs about knowledge (Muis, 2008; Chevrier et al., 2020). Procedural understanding, therefore, can be understood as a form of epistemic compromise rather than a transitional deficiency.

By contrast, the rise of fragmented responses reveals a more fundamental disruption. The errors observed are neither random nor incoherent; rather, they exhibit internal consistency grounded in structurally inappropriate assumptions. Students do not cease to reason, but instead reason within frameworks that fail to correspond to the relational structure of mathematical expressions. This form of misdirected coherence suggests that errors should be interpreted as indicators of epistemic positioning rather than simple misconceptions (Alvidrez et al., 2024; Richter & Schmid, 2010; Sarwadi & Shahrill, 2014). From this perspective, incorrect responses provide insight into how knowledge is constructed, stabilised, and, in some cases, misaligned with institutional expectations.

The emergence of such fragmentation can be further understood through the lens of the Anthropological Theory of the Didactic, particularly the notion of didactical transposition. The transformation of mathematical knowledge from scholarly forms into teachable content often entails reductions that compromise conceptual coherence (Abou-Hayt, 2024; Robutti, 2020; Suryadi & Priatna, 2021). When instructional tasks privilege procedural accessibility without maintaining structural integrity, students are more likely to develop segmented forms of knowledge. Under conditions of increased complexity, these segmented structures prove insufficient, leading to a breakdown in epistemic coordination. In a broader sense, this phenomenon resonates with concerns about epistemic injustice, where access to more coherent and powerful forms of knowledge is unevenly distributed within educational contexts (Tanswell & Rittberg, 2020).

The three-level model identified in this study—integrated, procedural, and fragmented—offers a way of conceptualising understanding as a continuum of epistemic alignment. These levels do not represent fixed categories but reflect shifts in how knowledge is organised and validated in response to task demands. In this regard, the distinction between concept image and concept definition (Tall & Vinner, 1981; Vinner, 1983; Rösken & Rolka, 2007) acquires a situational dimension, as discrepancies between the two become increasingly pronounced under conditions requiring coordinated reasoning. Moreover, the progression across tasks suggests that structural complexity operates as a diagnostic gradient of epistemic stability. As the coordination demands increase, only those students whose understanding is structurally integrated are able to maintain coherence. This observation aligns with studies on epistemic actions and classroom interaction, which emphasise that mathematical understanding is shaped through participation in practices of justification, validation, and meaning-making (Hwang et al., 2023; Ingram, 2020; Nachowitz, 2019). From this perspective, learning mathematics involves not only acquiring procedures but engaging in processes through which knowledge is legitimised and negotiated (Yamaguchi, 2025).

Methodologically, the use of Didactical Design Research enables close examination of these dynamics by foregrounding the relationship between task design and students' reasoning. The structured sequence of tasks makes it possible to observe how understanding is constructed, destabilised, and reorganised across different levels of complexity. Such an approach is particularly suited to investigating epistemic processes that are not readily captured through static or purely quantitative measures (Swan, 2020). These findings contribute to a broader shift toward viewing mathematical learning as an epistemic practice that extends beyond individual cognition. Understanding is not confined to internal mental representations but is constituted through interactions with tasks, representations, and institutional norms (Chiu et al., 2026; Vold & Schlimm, 2020; Mulligan, 2015). In this sense, meaningful learning depends on the capacity to coordinate multiple dimensions—conceptual, procedural, and contextual—within a coherent and adaptable structure of reasoning.

CONCLUSION

This study set out to examine how students' understanding of rational inequalities develops as a function of epistemic alignment under increasing levels of structural complexity. The findings demonstrate that students' reasoning cannot be adequately explained through a binary distinction between correct and incorrect responses. Instead, understanding emerges as a dynamic, situated configuration shaped by the interaction among conceptual knowledge, procedural actions, and task structure. A key contribution of this study lies in the identification of a three-level empirical model of epistemic alignment—integrated, procedural, and fragmented—which captures qualitative differences in how students organise and apply mathematical knowledge. Integrated understanding reflects epistemic coherence and transferability, procedural understanding represents a compensatory yet fragile form of reasoning, and fragmented understanding indicates structural breakdown and misalignment with formal mathematical knowledge. Importantly, these levels are not fixed categories, but positions within a continuum that shifts in response to task complexity. The study further demonstrates that task complexity functions as a diagnostic mechanism, revealing the stability and coherence of students' reasoning. As the demand for multi-layer conceptual coordination increases, students' understanding becomes increasingly vulnerable to disruption, leading to procedural reliance or structural breakdown. This finding underscores that meaningful learning depends not only on acquiring knowledge, but on the ability to integrate and coordinate multiple conceptual dimensions in real time.

Theoretically, this study advances a beyond-cognition perspective by positioning learning as an epistemic process that integrates cognitive, instructional, and institutional dimensions. It extends existing frameworks, such as the Theory of Didactical Situations and the Anthropological Theory of the Didactic, by conceptualizing institutionalization as a dynamic, variable mechanism rather than a fixed instructional phase. Practically, the findings highlight the importance of designing instructional tasks that support conceptual–procedural integration and promote epistemic coherence. Educators should move beyond procedural teaching approaches and focus on facilitating students' ability to coordinate multiple representations and justifications within a unified reasoning structure. Future research is recommended to examine the applicability of this epistemic alignment model across mathematical domains and educational contexts, and to explore instructional interventions that support the development of stable, transferable understanding.

REFERENCES

Abakah, F., & Brijlall, D. (2024). Misconceptions of mathematical concepts vis-à-vis how they pose as barriers to developing students' conceptual understanding. *Gulf Journal of Mathematics, 16*(2), 81–99. https://doi.org/10.56947/gjom.v16i2.1871

Abou-Hayt, I. (2024). A critical look at the anthropological theory of the didactic. *Research in Mathematics Education*, 1–22. https://doi.org/10.1080/14794802.2024.2344754

Alvidrez, M., Louie, N., & Tchoshanov, M. (2024). From mistakes, we learn? Mathematics teachers' epistemological and positional framing of mistakes. *Journal of Mathematics Teacher Education, 27*, 111–136. https://doi.org/10.1007/s10857-022-09553-4

Bahn, J., & Winsløw, C. (2019). Doing and investigating lesson study with the theory of didactical situations. In R. Huang, A. Takahashi, & J. P. da Ponte (Eds.), *Theory and practice of lesson study in mathematics* (Advances in Mathematics Education). Springer. https://doi.org/10.1007/978-3-030-04031-4_5

Brown, T., & Clarke, D. (2012). Institutional contexts for research in mathematics education. In M. Clements, A. Bishop, C. Keitel, J. Kilpatrick, & F. Leung (Eds.), *Third international handbook of mathematics education* (Vol. 27). Springer. https://doi.org/10.1007/978-1-4614-4684-2_15

Castela, C., & Romo-Vázquez, A. (2022). Towards an institutional epistemology. In R. Biehler, M. Liebendörfer, G. Gueudet, C. Rasmussen, & C. Winsløw (Eds.), *Practice-oriented research in tertiary mathematics education*. Springer. https://doi.org/10.1007/978-3-031-14175-1_30

Chevrier, M., Muis, K. R., & Di Leo, I. (2020). Calibration to task complexity: The role of epistemic cognition. *The Journal of Experimental Education*, 88(1), 1–26. https://doi.org/10.1080/00220973.2019.1584740

Chiu, M. S., Lin, F. L., & Yang, K. L. (2026). Beyond cognition: The four-seasons framework for integrating affect into mathematics teaching. *Mathematics Education Research Journal*. https://doi.org/10.1007/s13394-026-00561-8

Eynde, P. O., De Corte, E., & Verschaffel, L. (2006). Accepting emotional complexity: A socio-constructivist perspective on the role of emotions in the mathematics classroom. *Educational Studies in Mathematics, 63*, 193–207. https://doi.org/10.1007/s10649-006-9034-4

Gravemeijer, K. (2020). A socio-constructivist elaboration of realistic mathematics education. In M. Van den Heuvel-Panhuizen (Ed.), *National reflections on the Netherlands didactics of mathematics* (ICME-13 Monographs). Springer. https://doi.org/10.1007/978-3-030-33824-4_12

Greeno, J. G. (1978). Understanding and procedural knowledge in mathematics instruction. *Educational Psychologist, 12*(3), 262–283. https://doi.org/10.1080/00461527809529180

Greeno, J. G. (1978). Understanding and procedural knowledge in mathematics instruction. *Educational Psychologist*, 12(3), 262–283. https://doi.org/10.1080/00461527809529180

Heyd-Metzuyanim, E., Munter, C., & Greeno, J. (2018). Conflicting frames: A case of misalignment between professional development efforts and a teacher's practice in a high school mathematics classroom. *Educational Studies in Mathematics*, 97, 21–37. https://doi.org/10.1007/s10649-017-9777-0

Hodgen, J. (2011). Knowing and identity: A situated theory of mathematics knowledge in teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical knowledge in teaching* (Mathematics Education Library, Vol. 50). Springer. https://doi.org/10.1007/978-90-481-9766-8_3

Hoth, J., Larrain, M., & Kaiser, G. (2022). Identifying and dealing with student errors in the mathematics classroom: Cognitive and motivational requirements. *Frontiers in Psychology, 13*, 1057730. https://doi.org/10.3389/fpsyg.2022.1057730

Hudson, B. (2018). Powerful knowledge and epistemic quality in school mathematics. *London Review of Education*, 16(3), 384–397. https://doi.org/10.18546/LRE.16.3.03

Hwang, J., Choi, K. M., & Hand, B. (2023). Epistemic actions and mathematics achievement. *International Journal of Science and Mathematics Education*, 21, 787–809. https://doi.org/10.1007/s10763-022-10278-2

Ingram, J. (2020). Epistemic management in mathematics classroom interactions: Student claims of not knowing or not understanding. *The Journal of Mathematical Behavior*, 58, 100754. https://doi.org/10.1016/j.jmathb.2019.100754

Löfström, E., & Pursiainen, T. (2015). Knowledge and knowing in mathematics and pedagogy: A case study of mathematics student teachers' epistemological beliefs. *Teachers and Teaching*, 21(5), 527–542. https://doi.org/10.1080/13540602.2014.995476

Mair, C., & Li, R. (2025). Epistemic limits of mathematical sciences in global higher education. *Globalisation, Societies and Education*, 1–13. https://doi.org/10.1080/14767724.2025.2549720

McCormick, R. (1997). Conceptual and procedural knowledge. *International Journal of Technology and Design Education*, 7, 141–159. https://doi.org/10.1023/A:1008819912213

Muis, K. R. (2008). Epistemic profiles and self-regulated learning: Examining relations in the context of mathematics problem solving. *Contemporary Educational Psychology*, 33(2), 177–208. https://doi.org/10.1016/j.cedpsych.2006.10.012

Mulligan, J. (2015). Looking within and beyond the geometry curriculum: Connecting spatial reasoning to mathematics learning. *ZDM Mathematics Education*, 47, 511–517. https://doi.org/10.1007/s11858-015-0696-1

Nachowitz, M. (2019). Intent and enactment: Writing in mathematics for conceptual understanding. *Investigations in Mathematics Learning*, 11(4), 245–257. https://doi.org/10.1080/19477503.2018.1461051

Novotná, J., & Hošpesová, A. (2022). Theory of didactical situations in mathematics as one of the theoretical approaches to the LPS analysis of classroom data: The case of institutionalization. *ZDM – Mathematics Education, 54*, 303–316. https://doi.org/10.1007/s11858-022-01331-z

Richter, T., & Schmid, S. (2010). Epistemological beliefs and epistemic strategies in self-regulated learning. *Metacognition and Learning, 5*, 47–65. https://doi.org/10.1007/s11409-009-9038-4

Robutti, O. (2020). Meta-didactical transposition. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer. https://doi.org/10.1007/978-3-030-15789-0_100012

Rösken, B., & Rolka, K. (2007). Integrating intuition: The role of concept image and concept definition for students' learning of integral calculus. In *Beliefs and mathematics*. https://doi.org/10.1108/book-978-1-60752-611-720251017

Sarwadi, H. R. H., & Shahrill, M. (2014). Understanding students' mathematical errors and misconceptions: The case of year 11 repeating students. *Mathematics Education Trends and Research*, 2014, 1–10. https://doi.org/10.5899/2014/metr-00051

Suryadi, D., & Priatna, N. (2021). Analysis of didactic transposition and HLT as a rationale in designing didactic situation. In *Proceedings of the 4th Sriwijaya University Learning and Education International Conference (SULE-IC 2020)* (pp. 567–574). Atlantis Press. https://doi.org/10.2991/assehr.k.201230.164

Swan, M. (2020). Design research in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer. https://doi.org/10.1007/978-3-030-15789-0_180

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151–169. https://doi.org/10.1007/BF00305619

Tanswell, F. S., & Rittberg, C. J. (2020). Epistemic injustice in mathematics education. *ZDM Mathematics Education*, 52, 1199–1210. https://doi.org/10.1007/s11858-020-01174-6

Usó-Doménech, J. L., Nescolarde-Selva, J. A., & Gash, H. (2022). Epistemological considerations about mathematical concepts. *Kybernetes, 51*(1), 95–115. https://doi.org/10.1108/K-07-2020-0485

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology*, 14(3), 293–305. https://doi.org/10.1080/0020739830140305

Vold, K., & Schlimm, D. (2020). Extended mathematical cognition: External representations with non-derived content. *Synthese*, 197, 3757–3777. https://doi.org/10.1007/s11229-019-02097-w

Yamaguchi, J. A. R. (2025). Voice to validation: An epistemic-legitimation cycle for pluriversal mathematics education. *Policy Futures in Education*, 23(8), 1468–1489. https://doi.org/10.1177/14782103251367219